PLANE TEMPERATURE FIELD IN AN INFINITE SOLID WITH A CYLINDRICAL FOREIGN INCLUSION

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An examination is made of the problem of determining the steady temperature field in a medium with a cylindrical inclusion, separated from the medium by a thin intermediate layer, for an assigned steady temperature at infinity. The problem is reduced to solution of a system of singular integro-differential equations, which are also valid for a solid containing an inclusion in the form of a thin non-closed cylindrical shell.

Let there be an infinite homogeneous solid containing a cylindrical inclusion, separated from the solid by a thin intermediate layer of constant thickness. We shall make the z axis of a rectangular system of coordinates coincide with the axis of the inclusion.

We shall examine the problem of determining the steady temperature field T(x, y) in this kind of system, under the assumption that at a large enough distance from the inclusion the temperature distribution is described by a given harmonic function $t_\infty(x, y)$.

To simplify the problem we shall replace the intermediate by some physical surface with thermophysical properties as shown [1, 2]. The intersection of this cylindrical surface with the plane xOy gives a certain contour L (see figure), on which the following conditions must be satisfied [1, 2]:

$$\lambda_0 \frac{\partial^2}{\partial s_0^2}(T^+ + T^-) + 2\left(\lambda_1 \frac{\partial T^+}{\partial n_0} - \lambda_2 \frac{\partial T^-}{\partial n_0}\right) = 0,$$

$$\lambda_0 \frac{\partial^2}{\partial s_0^2}(T^+ - T^-) + 6\left(\lambda_1 \frac{\partial T^+}{\partial n_0} + \lambda_2 \frac{\partial T^-}{\partial n_0}\right) - 12h(T^+ - T^-) = 0. \quad (1)$$

In what follows we shall regard the contour L as a Lyapunov line.

The desired temperature T(x, y) may of course be represented in the form

$$T(x, y) = t(x, y) + t_\infty(x, y), \quad (2)$$

where t is the temperature due to the perturbation of the given temperature field due to the presence of the inclusion in the solid. We shall assume that at infinity $t(x, y) \to 0$.

To determine the harmonic inside and outside contour L of the function t(x, y), which will vanish at infinity, we obtain, because of relation (1), the following boundary conditions on L:

$$\lambda_0 \frac{\partial^2}{\partial s_0^2}(t^+ + t^-) + 2\left(\lambda_1 \frac{\partial t^+}{\partial n_0} - \lambda_2 \frac{\partial t^-}{\partial n_0}\right) = f_1(s_0),$$

$$\lambda_0 \frac{\partial^2}{\partial s_0^2}(t^+ - t^-) + 6\left(\lambda_1 \frac{\partial t^+}{\partial n_0} + \lambda_2 \frac{\partial t^-}{\partial n_0}\right) - 12h(t^+ - t^-) = f_2(s_0), \quad (3)$$

where

$$f_1(s_0) = -2\left[\lambda_0 \frac{\partial^2 t_\infty}{\partial s_0^2} + (\lambda_1 - \lambda_2)\frac{\partial t_\infty}{\partial n_0}\right],$$

$$f_2(s_0) = -6(\lambda_1 + \lambda_2)\frac{\partial t_\infty}{\partial n_0}. \quad (4)$$

We shall seek the function t(x, y) in the form of a sum of logarithmic potentials of a simple layer and a double layer with the respective densities $\gamma(s)$ and p(s):

$$t(x, y) = u(x, y) + v(x, y), \quad (5)$$

$$u(x, y) = \frac{1}{2\pi}\int_L \gamma(s)\frac{\cos(r, n)}{r}ds,$$

$$v(x, y) = \frac{1}{2\pi}\int_L p(s)\ln\frac{1}{r}ds. \quad (6)$$

It is well known [3] that the potential of the double layer tends to zero at infinity; the potential of the simple layer has a logarithmic singularity at infinity, and will vanish there only if the entire "mass" of the simple layer is equal to zero:

$$\int_L p(s)\,ds = 0. \quad (7)$$

The limiting values of the potentials on contour L have the form [3]

$$u^\pm(s_0) = \pm\frac{1}{2}\gamma(s_0) + \frac{1}{2\pi}\int_L \gamma(s)\frac{\sin\alpha}{r}ds,$$

$$v^\pm(s_0) = \frac{1}{2\pi}\int_L p(s)\ln\frac{1}{r}ds. \quad (8)$$

To determine the normal and the second tangential derivatives of the potentials, it is convenient to use an integral of the Cauchy type, which is closely connected with the potentials of the simple and double layers.

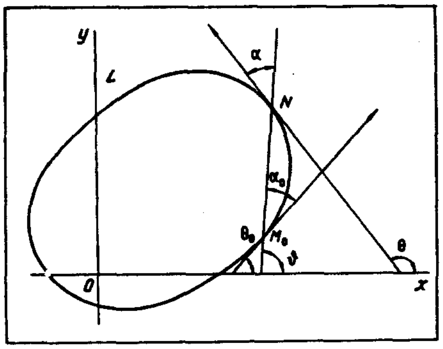
Let

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\mu(\zeta)\,d\zeta}{\zeta - z} \quad (9)$$

be an integral of the Cauchy type, whose density $\mu(\zeta)$ is the actual function satisfying the Hölder condition. As is known [4], the real part of the integral is the potential of the double layer with density $\mu(s)$, while the imaginary part is the potential of the simple layer with density $p(s) = -d\mu/ds$.

It is evident that the derivatives of potentials (6) will also be equal to the real and imaginary parts of the corresponding derivative of the integral (9) of the

Cauchy type. Then we must replace $\mu(s)$ by $\gamma(s)$ under the integrals of the real part, and put $\mu'(s) = -p(s)$ under the integrals of the imaginary part.



Schematic of the problem.

The limiting values of the normal and tangential derivatives of the Cauchy type integral are expressed in terms of the limiting values of its derivative with respect to the complex coordinate. Then, if $\mu^{(m)}(\zeta)$ satisfies the Hölder condition, the limiting values of the m-th order derivative of the Cauchy type integral are determined according to the formula [5]:

$$[\Phi^{(m)}(\zeta_0)]^{\pm} = \pm\frac{1}{2}\mu^{(m)}(\zeta_0) + \frac{1}{2\pi i}\int_L \frac{\mu^{(m)}(\zeta)}{\zeta-\zeta_0}\,d\zeta. \quad (10)$$

We shall find the normal derivatives of the potentials (6).

We have

$$\left(\frac{d\Phi}{dn_0}\right)^{\pm} = \left(\frac{d\Phi}{d\zeta_0}\right)^{\pm}\frac{d\zeta_0}{dn_0}. \quad (11)$$

Substituting the expression $[\Phi'(\zeta_0)]^{\pm}$ from (10) into (11), and taking into account that

$$\frac{d\zeta_0}{dn_0} = ie^{i\Theta_0}, \quad \frac{1}{\zeta-\zeta_0} = \frac{\exp(-i\vartheta)}{r}, \quad \vartheta-\Theta_0 = \alpha_0,$$

we obtain

$$\left(\frac{d\Phi}{dn_0}\right)^{\pm} = \pm\frac{i}{2}\mu'(s_0) + \frac{1}{2\pi}\int_L \mu'(s)\,\frac{\exp(-i\alpha_0)}{r}\,ds. \quad (12)$$

Separating the real and imaginary parts, with appropriate substitution of the densities, we find

$$\frac{du^{\pm}}{dn_0} = \frac{1}{2\pi}\int_L \gamma'(s)\,\frac{\cos\alpha_0}{r}\,ds,$$

$$\frac{dv^{\pm}}{dn_0} = \mp\frac{1}{2}p(s_0) + \frac{1}{2\pi}\int_L p(s)\,\frac{\sin\alpha_0}{r}\,ds. \quad (13)$$

We should understand the integral in the first part of (13) in the sense of the principal value according to Cauchy.

The second derivative of the Cauchy type integral with respect to s_0 is expressed in terms of derivatives with respect to the complex coordinate of the contour ζ_0 according to the formula

$$\frac{d^2\Phi}{ds_0^2} = \frac{d^2\Phi}{d\zeta_0^2}\left(\frac{d\zeta_0}{ds_0}\right)^2 + \frac{d\Phi}{d\zeta_0}\,\frac{d^2\zeta_0}{ds_0^2}. \quad (14)$$

From (10) we shall find the limiting values $\Phi'(\zeta_0)$ and $\Phi''(\zeta_0)$, expressed as functions of the arc of the contour. Bearing in mind that $d\zeta = e^{i\Theta}ds$, we have

$$[\Phi'(\zeta_0)]^{\pm} = \pm\frac{1}{2}\mu'(s_0)\exp(-i\Theta_0) + \frac{1}{2\pi i}\int_L \mu'(s)\,\frac{\exp(-i\vartheta)}{r}\,ds,$$

$$[\Phi''(\zeta_0)]^{\pm} = \frac{1}{2}[\pm\mu''(s_0) - iK_0\mu'(s_0)]\exp(-2i\Theta_0) + \frac{1}{2\pi i}\int_L [\mu''(s) - iK\mu'(s)]\,\frac{\exp[-i(\vartheta+\Theta)]}{r}\,ds. \quad (15)$$

Substituting (15) into (14), and taking into account that $\Theta = \vartheta + \alpha$, we obtain

$$\left[\frac{d^2\Phi}{ds_0^2}\right]^{\pm} = \pm\frac{1}{2}\mu''(s_0) + \frac{K_0}{2\pi}\int_L \mu'(s)\,\frac{\exp(-i\alpha_0)}{r}\,ds + \frac{1}{2\pi i}\int_L [\mu''(s) - iK\mu'(s)]\,\frac{\exp[-i(2\alpha_0+\alpha)]}{r}\,ds. \quad (16)$$

From (16), following separation of the real and imaginary parts, we obtain

$$\frac{d^2u^{\pm}}{ds_0^2} = \pm\gamma''(s_0) - \frac{1}{2\pi}\int_L \gamma''(s)\,\frac{\sin(2\alpha_0+\alpha)}{r}\,ds - \frac{1}{2\pi}\int_L K\gamma'(s)\,\frac{\cos(2\alpha_0+\alpha)}{r}\,ds + \frac{K_0}{2\pi}\int_L \gamma'(s)\,\frac{\cos\alpha_0}{r}\,ds, \quad (17)$$

$$\frac{d^2v^{\pm}}{ds_0^2} = \frac{1}{2\pi}\int_L p'(s)\,\frac{\cos(2\alpha_0+\alpha)}{r}\,ds - \frac{1}{2\pi}\int_L K\,p(s)\,\frac{\sin(2\alpha_0+\alpha)}{r}\,ds + \frac{K_0}{2\pi}\int_L p(s)\,\frac{\sin\alpha_0}{r}\,ds. \quad (18)$$

We interpret the last two integrals in (17), and the first integral in (18), in the sense of the principal value according to Cauchy.

Substituting (8), (13), (17) and (18), taking account of (5), into the boundary conditions (3), we obtain the following system of singular integro-differential equations to determine $\gamma(s)$ and $p(s)$:

$$\frac{\lambda_0K_0+\lambda_1-\lambda_2}{\pi}\int_L\left[\gamma'(s)\,\frac{\cos\alpha_0}{r} + p(s)\,\frac{\sin\alpha_0}{r}\right]ds - \quad (19)$$

$$\frac{\lambda_0}{\pi}\int_L [\gamma''(s) + Kp(s)]\,\frac{\sin(2\alpha_0+\alpha)}{r}\,ds - (\lambda_1+\lambda_2)\,p(s_0) -$$

$$-\frac{\lambda_0}{\pi}\int\limits_L [K\gamma'(s)-p'(s)]\,\frac{\cos(2\alpha_0+\alpha)}{r}\,ds=f_1(s_0);$$

$$\lambda_0\gamma''(s_0)-12h\gamma(s_0)-3(\lambda_1-\lambda_2)\,p(s_0)+\frac{3(\lambda_1+\lambda_2)}{\pi}\times$$

$$\times\int\limits_L\left[\gamma'(s)\,\frac{\cos\alpha_0}{r}+p(s)\,\frac{\sin\alpha_0}{r}\right]ds=f_2(s_0). \qquad (19)$$
(cont'd)

Equations (19) were obtained for closed contours. They are true also for open contours, with the proviso that at the ends of the contour, $\gamma(s)$, $\gamma'(s)$ and $p(s)$ are equal to zero. This condition indicates the finiteness of the potentials and of their derivatives at the ends.

Having determined $\gamma(s)$ and $p(s)$ from the system of Eqs. (19), we may find, from formulas (6) and (5), the function $t(x, y)$ which describes the perturbation of the temperature field in the vicinity of the inclusion.

We note that it is convenient to determine the potentials $u(x, y)$ and $v(x, y)$, not from (6), but with the help of the Cauchy type integral (9), making use of the fact that, as has already been noted above, the real part of $\Phi(z)$ is the potential of the double layer with density $\gamma(s) = \mu(s)$, while the imaginary part is the potential of the simple layer $v(x, y)$ with density $p(s) = -\mu'(s)$.

We shall examine some special cases.

If there is perfect thermal contact between the solid and the inclusion ($\lambda_0 = 0$, $h = \infty$), from the second equation of (19) we obtain that $\gamma(s) = 0$, while from the first equation—the Fredholm integral equation of the second kind

$$p(s_0)-\frac{\lambda_1-\lambda_2}{\pi(\lambda_1+\lambda_2)}\int\limits_L p(s)\,\frac{\sin\alpha_0}{r}\,ds=2\,\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}\,\frac{\partial t_\infty}{\partial n_0}. \qquad (20)$$

In the case when L is a section of the real axis $x \le l$ (we have a crack in the xOy plane, between the edges of which there is imperfect thermal contact), then, using $\lambda_1 = \lambda_2 = \lambda$, $K_0 = K = 0$, and taking into account that

$$\sin\alpha_0/r=\sin\alpha/r=0,\ \cos\alpha_0/r=\cos\alpha/r=1/(\xi-x),$$

we find

$$\lambda_0\gamma''(x)-12h\gamma(x)+\frac{6\lambda}{\pi}\int\limits_{-l}^{l}\frac{\gamma'(\xi)\,d\xi}{\xi-x}=-12\lambda\,\frac{\partial t_\infty(x,0)}{\partial y},$$

$$\lambda p(x)-\frac{\lambda_0}{2\pi}\int\limits_{-l}^{l}\frac{p'(\xi)\,d\xi}{\xi-x}=\lambda_0\,\frac{\partial^2 t_\infty(x,0)}{\partial x^2}. \qquad (21)$$

If the thermal conductivity of the crack is $\lambda_0 = 0$, then $p(x) = 0$, and to determine $\gamma(x)$ we obtain the Prandtl integro-differential equation

$$h\gamma(x)-\frac{\lambda}{2\pi}\int\limits_{-l}^{l}\frac{\gamma'(\xi)\,d\xi}{\xi-x}=\lambda\,\frac{\partial t_\infty(x,0)}{\partial y}. \qquad (22)$$

There is at present no exact solution of this equation. It may be reduced to a regular Fredholm equation. For numerical calculations, however, it is convenient to use one of the approximate methods, for example, the method of trigonometrical expansions [6], or the method of Multhopp [7, 8].

Finally, we shall find the temperature field in a plane with a circular inclusion of radius R, when the uniform temperature field,

$$t_\infty=a+b\rho\cos\varphi, \qquad (23)$$

is given at infinity, ρ and φ being polar coordinates (the origin of the polar system of coordinates has been chosen to be at the center of the circle, while the polar axis is directed parallel to the heat flux at infinity).

Putting

$$K=K_0=\frac{1}{R},\quad \alpha=\alpha_0=\frac{\varphi-\varphi_0}{2},\quad r=2R\sin\frac{\varphi-\varphi_0}{2},$$

$$f_1=2b\left(\frac{\lambda_0}{R}+\lambda_1-\lambda_2\right)\cos\varphi_0,\quad f_2=6b(\lambda_1+\lambda_2)\cos\varphi_0$$

in (19), integrating by parts, and taking into account condition (7) and the continuity of $p(\varphi)$, $\gamma(\varphi)$ and $\gamma'(\varphi)$, we obtain

$$\frac{1}{2\pi R}\int\limits_0^{2\pi}[(\lambda_1-\lambda_2)\gamma'(\varphi)+\lambda_0 p'(\varphi)]\ \mathrm{ctg}\,\frac{\varphi-\varphi_0}{2}\,d\varphi-$$

$$-(\lambda_1+\lambda_2)\,p(\varphi_0)=$$

$$=2b\left(\frac{\lambda_0}{R}+\lambda_1-\lambda_2\right)\cos\varphi_0, \qquad (24)$$

$$\frac{\lambda_0}{R^2}\gamma''(\varphi_0)-12h\gamma(\varphi_0)-3(\lambda_1-\lambda_2)\,p(\varphi_0)+$$

$$+\frac{3(\lambda_1+\lambda_2)}{2\pi R}\int\limits_0^{2\pi}\gamma'(\varphi)\ \mathrm{ctg}\,\frac{\varphi-\varphi_0}{2}\,d\varphi=6b(\lambda_1+\lambda_2)\cos\varphi_0.$$

We shall seek a solution of (24) in the form

$$\gamma(\varphi)=A\cos\varphi,\quad p(\varphi)=B\cos\varphi. \qquad (25)$$

Following evaluation of the integrals, we find

$$A=-\frac{12bR^2(\lambda_0\lambda_2+2R\lambda_1\lambda_2)}{\lambda_0^2+4R(\lambda_1+\lambda_2)(\lambda_0+3hR^2)+12R^2(h\lambda_0+\lambda_1\lambda_2)},$$

$$B=$$

$$=-\frac{2b\left[\lambda_0^2+3R\lambda_0(\lambda_1+\lambda_2+4hR)+R(\lambda_1-\lambda_2)(\lambda_0+12hR^2)\right]}{\lambda_0^2+4R(\lambda_1+\lambda_2)(\lambda_0+3hR^2)+12R^2(h\lambda_0+\lambda_1\lambda_2)}.$$

Denoting the complex coordinate of a point of the circle by ζ, we may represent $\gamma(s)$ in the complex form

$$\gamma(\zeta)=\frac{A}{2R}(\zeta+\bar\zeta)=\frac{A}{2R}\left(\zeta+\frac{R^2}{\zeta}\right).$$

Substituting this value of γ into (9) in place of μ, and separating the real part, we obtain

$$u(\rho, \varphi) = \operatorname{Re}\frac{Az}{2R} = \frac{A\rho}{2R}\cos\varphi \qquad (\rho < R),$$

$$u(\rho, \varphi) = -\operatorname{Re}\frac{AR}{2z} = -\frac{AR}{2\rho}\cos\varphi \qquad (\rho > R).$$

Further, putting

$$\mu = -\int p(s)\,ds = -BR\sin\varphi = -\frac{B}{2i}\left(\zeta - \frac{R^2}{\zeta}\right)$$

in (9), and separating the imaginary part, we find

$$v(\rho, \varphi) = -\operatorname{Im}\frac{Bz}{2i} = \frac{B\rho}{2}\cos\varphi \qquad (\rho < R),$$

$$v(\rho, \varphi) = -\operatorname{Im}\frac{BR^2}{2iz} = \frac{BR^2}{2\rho}\cos\varphi \qquad (\rho > R).$$

For the temperature of the inclusion and of the solid, on the basis of (23), (5), and (2), we obtain, respectively,

$$T = a + \left(b + \frac{A + BR}{2R}\right)\rho\cos\varphi \qquad (\rho < R),$$

$$T = a + \left(b\rho - \frac{A - BR}{2}\,\frac{R}{\rho}\right)\cos\varphi \qquad (\rho > R).$$

The last example was also examined in reference [2].

NOTATION

$T(x, y)$ denotes temperature; $t_\infty(x, y)$ is temperature at infinity; $t(x, y)$ is a function describing the perturbation of the temperature field in the vicinity of the intermediate layer; λ_0 is reduced thermal conductivity of the layer; h is its thermal conductivity; λ_1, λ_2 are thermal conductivities of the inclusion and of the solid; s, s_0 are the length of arc coordinates of points N and M_0 of the contour L; n_0 is the inside normal to L at the point M_0; $u(x, y)$, $v(x, y)$ are the logarithmic potentials of the double and simple layers; $\gamma(s)$, $p(s)$ are the densities of the double and simple layers; $\mu(\zeta)$ is the density of the Cauchy type integral; r is the distance between the two points; α, α_0 are the angles between the vector $\overrightarrow{M_0N}$ and the positive tangents to L at the points N and M_0, respectively; θ is the angle between the axis Ox and the vector $\overrightarrow{M_0N}$; Θ, Θ_0 are angles between the axis Ox and the positive tangents at the points N and M_0, respectively; K, K_0 are the curvature of the contour L at the points N and M_0; a, b are given constants, determining the temperature at infinity. The subscripts plus and minus denote the limiting values of the quantities when approaching the contour L from the side of the inclusion and of the solid, respectively.

REFERENCES:

1. Ya. S. Pidstrygach, DAN URSR, no. 7, 1963.
2. Ya. S. Podstrigach, IFZh, 6, no. 10, 1963.
3. L. N. Sretenskii, Theory of the Non-Newtonian Potential [in Russian], OGIZ, 1946.
4. N. I. Muskhelishvili, Singular Integral Equations [in Russian], Fizmatgiz, 1962.
5. F. D. Gakhov, Boundary Problems [in Russian], Fizmatgiz, 1963.
6. V. V. Golubev, Lectures on Airfoil Theory [in Russian], GITTL, 1949.
7. A. I. Kalandiya, Matem. sb., 42, no. 2, 1957.
8. H. Multhopp, Luftfahrtforschung, 15, 153, 1938.

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